Classification of 5-dimensional MD-algebras having commutative derived ideals

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Abstract

In this paper, we study a subclass of the class of MD-algebras, i.e., the class of solvable real Lie algebras such that the K-orbits of its corresponding connected and simply connected Lie groups are either orbits of dimension zero or orbits with maximal dimensions. Our main result is to classify, up to isomorphism, all the 5-dimensional MD-algebras having commutative derived ideals.

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Introduction

The concept of C*-algebras was first introduced by Gelfand and Naimark in 1943. It is well known that C*-algebras can be applied to mathematics, mechanics and physics, however, the problem of describing the structure of C*-algebras, in general, is still open.

The method of describing the structure of C*-algebras by using K-functors was first suggested by D. N. Diep ([2]) in 1974. By applying the K-homology functors proposed by Brown - Douglas - Fillmore (for brevity, the BDF K-functors), Diep gave a description for the C*(Aff$\mathbb{R}$) of the group Aff$\mathbb{R}$ of the affine transformations of the real line. In 1975, by using the method of Diep, J. Rosenberg ([7], [8]) gave a description for the C*-algebra of the group Aff$\mathbb{C}$ and some other groups. In 1977, D.N.Diep ([3]) further gave a complete system of invariants of C*-algebras of type I by using the BDF K-homology functors. Hence, it is natural to propose the following two general problems:

- Generalize the K-homology functors so that these functors can be applied to a larger class of C*-algebras.

- Find the C*-algebras which can be described by using the generalized K-functors.

Concerning the first problem, we note that G. G. Kasparov ([5]) in 1980 introduced the concept of KK-functors which is a generalized concept of BDF K-homology functors. Then by using KK-functors, G.G. Kasparov described the C*-algebra of the Heisenberg groups H_{2n+1}.

For the second problem, it was noticed that this problem is closely related with the Orbit Method proposed by A.A. Kirillov ([6]) in 1962. After studying the Kirillov's Orbit Method, Diep in 1980 suggested to consider the class of Lie groups and Lie algebras MD and $\overline{MD}$ ([4]) so that the C*-algebras of them can be described by using KK-functors. If G is an n-dimensional real Lie group, then G is called a MDn-group or a MD-group with dimension n iff the orbits of G in the K-representation (K-orbits) are orbits of dimension zero or orbits of maximal dimension (i.e. dimension k, where k is some even constant, $k \leq n$). When k = n, we call G an $\overline{MDn}$-group or $\overline{MD}$-group of dimension n. The corresponding Lie algebra Lie(G) of G is said to be an MDn-algebra or $\overline{MDn}$-algebra, respectively. It is clear that the class $\overline{MD}$ is a subclass of the class MD. Thus, the problem of classifying MD-algebras, describing the K-representation of MD-groups and characterizing the C*-algebras of MD-groups is significant. Note that all the Lie algebras and the Lie groups of dimension n with $n < 4$ are MD-algebras and MD-groups, and moreover they can be listed easily. So we only take interest in MDn-groups and MDn-algebras for $n \geq 4$.

We remark here that all $\overline{MD}$-algebras (of arbitrary dimension) was classified, up to isomorphism, by H. H. Viet in [9]. This class includes only the following algebras:

- $\mathbb{R}^n$ - The commutative Lie Algebra of dimension n;

- Lie(Aff$\mathbb{R}$) - The Lie algebra of the group of affine traformations of the real straight line;

- Lie(Aff$\mathbb{C}$) - The Lie algebra of the group of affine transformations of the complex straight line.

It is noteworthy that Viet [9] also described the C*-algebras of the universal covering of group Aff$\mathbb{C}$ by using KK-functors. Thus, the C*-algebras of all groups of the class $\overline{MD}$ were described by Diep, Rosenberg and Viet.

The problem for the class of MD-algebras is much more complicated than $\overline{MD}$-algebras. In 1984, Dao Van Tra [11] listed all MD4-algebras. In 1990, all MD4-algebras were classified, up to isomorphism, by Vu (see [12], [13], [14]). Until quite recently, Vu together with Nguyen Cong Tri, Duong Minh Thanh and Duong Quang Hoa introduced some MD5 - algebras and MD5 - groups (see [15], [16], [17], [18], [19], [20]). Until the present moment, there is no complete classification for MDn-algebras with $n \geq 5$.

On the other hand, by studying the foliated manifold, Connes ([1]) in 1982 proposed the notion of C*-algebras associated with a measured foliation. The following question naturally arises: Can we describe the Connes C*-algebras by using KK-functors? In fact, Torpe has shown in [10] that the KK-functors are very useful and effective to describe the structure of Connes C*-algebras associated with the Reeb foliations.

The other reason for studying the class MD is based on the following fact: if G is a certain MD-group, then the family of its K-orbits with maximal dimension forms a measured foliation. This foliation is called MD-foliation associated with G. Furthermore, the C*-algebra of G can be easily described when the Connes C*-algebra of MD-foliation associated with G is known. Hence, the problem of classifying the topology and describing the Connes C*-algebras of the class of MD-foliations is worth to study.

On this aspect, Vu in 1992 gave a topological classification of all MD4-foliations and described all Connes C*-algebras of them by using the KK-functors (see [12], [13], [14]). We noticed that the Connes C*-algebras of MDn-foliations with $n > 4$ has not yet been described. Following [9], if $\mathcal{G}$ is an MD-algebra then the second derived ideal $\mathcal{G}^2 = [\mathcal{G}^1, \mathcal{G}^1] = [[\mathcal{G}, \mathcal{G}], [\mathcal{G}, \mathcal{G}]]$ is commutative, however, the converse is not true. Therefore, we need to consider only $\mathcal{G}$ for which $\mathcal{G}^2$ is commutative. In particular, if $\mathcal{G}^2 = 0$ (i.e. $\mathcal{G}^1$ is commutative) then $\mathcal{G}$ could be an MD-algebra. Hence, we will restrict ourself only to this case. Our main result is to classify, up to an isomorphism, all MD5-algebras $\mathcal{G}$ having commutative derived ideal $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}]$. The topology of MD5-foliations associated with the MD5-groups and the description of Connes C*-algebras of these foliations will be considered and studied later on.

1 Preliminaries

We first recall in this Section some preliminary results and notations which will be used in the sequel. For more detailed information, the reader is referred to [4] and [6].

1.1 The co-adjoint Representation and K-orbits

Let G be a Lie group. Let $\mathcal{G} = \mathrm{Lie(G)}$ be the Lie algebra of G and we use $\mathcal{G}^*$ to denote the dual space of $\mathcal{G}$. For every g $\in$ G, we denote the internal automorphism associated with g by $A_{(g)}$, and whence, $A_{(g)} : G \longrightarrow G$ can be defined as follows

$$A_{(g)}(x) := g.x.g^{-1}, \forall x \in G.$$

The above automorphism induces the following mapping:

$$A_{(g)_*} : \mathcal{G} \longrightarrow \mathcal{G}$$

$$X \longmapsto A_{(g)_*}(X) := \frac{d}{dt}[g.exp(tX)g^{-1}] \mid_{t=0}$$

which is called *the tangent mapping* of $A_{(g)}$.

We now formulate the following definitions.

Definition 1.1.1. *The action*

$$Ad : G \longrightarrow Aut(\mathcal{G})$$

$$g \longmapsto Ad(g) := A_{(g)_*}$$

is called the adjoint representation of G in $\mathcal{G}$.

Definition 1.1.2. *The action*

$$K : G \longrightarrow Aut(\mathcal{G}^*)$$

$$g \longmapsto K_{(g)}$$

such that

$$\langle K_{(g)}F, X\rangle := \langle F, Ad(g^{-1})X\rangle; \quad (F \in \mathcal{G}^*,\, X \in \mathcal{G})$$

is called the co-adjoint representation or K-representation of G in $\mathcal{G}^*$.

Definition 1.1.3. *Each orbit of the co-adjoint representation of G is called a K-orbit of G.*

Thus, for every $F \in \mathcal{G}^*$, the K-orbit containing F defined above can be written by

$$\Omega_F := \{K_{(g)}F/g \in G\}.$$

The dimension of every K-orbit of an arbitrary Lie group G is always even. In order to define the dimension of the K-orbits Ω_F for each F from the dual space $\mathcal{G}^*$ of the Lie algebra $\mathcal{G} =$ Lie (G) of G, it is useful to consider the following skew-symmetric bilinear form B_F on $\mathcal{G}$

$$B_F(X,Y) := \langle F, [X,Y]\rangle; \, \forall\, X, Y \in \mathcal{G}.$$

Denote the stabilizer of F under the co-adjoint representation of G in $\mathcal{G}^*$ by G_F and $\mathcal{G}_F :=$ Lie(G_F).

We shall need in the sequel the following result.

Proposition 1.1.4 (see [6, Section 15.1]). *$KerB_F = \mathcal{G}_F$ and $dim\Omega_F = dim\mathcal{G} - dim\mathcal{G}_F$.* □

1.2 MDn-Groups and MDn-Algebras

Definition 1.2.1 (see [4, Chapter 4, definition 1.1]). *An MDn-group is an n-dimensional real solvable Lie group such that its K-orbits are orbits of dimension zero or maximal dimension. The Lie algebra of an MDn-group is called an MDn-algebra.*

The following proposition gives a necessary condition for a Lie algebra belonging to the class of all MD-algebras.

Proposition 1.2.2 (see [9, Theorem 4]). *Let $\mathcal{G}$ be an MD-algebra. Then its second derived ideal $\mathcal{G}^2 := [[\mathcal{G}, \mathcal{G}], [\mathcal{G}, \mathcal{G}]]$ is commutative.* $\square$

We point out here that the converse of the above result is in general not true. In other words, the above necessary condition is not a sufficient condition. We now only consider the 5-dimensional Lie algebras $\mathcal{G}$ having a second derived ideal $\mathcal{G}^2 = \{0\}$, i.e., the derived ideal $\mathcal{G}^1$ is commutative. Thus, the $\mathcal{G}$ could be an MD5-algebra.

2 The Main Result

From now on, we use $\mathcal{G}$ to denote an Lie algebra of dimension 5. We always choose a suitable basis $(X_1, X_2, X_3, X_4, X_5)$ in $\mathcal{G}$ so that $\mathcal{G}$ is isomorphic to $\mathbb{R}^5$ as a real vector space. The notation $\mathcal{G}^*$ will be used to denote the dual space of $\mathcal{G}$. Clearly, $\mathcal{G}^*$ can be identified with $\mathbb{R}^5$ by fixing in it the basis $(X_1^*, X_2^*, X_3^*, X_4^*, X_5^*)$ which is the dual of the basis $(X_1, X_2, X_3, X_4, X_5)$.

Theorem 2.1. *Let $\mathcal{G}$ be an MD5-algebra whose $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}]$ is commutative. Then the following assertions hold.*

I. If $\mathcal{G}$ is decomposable, then $\mathcal{G} \cong \mathcal{H} \oplus \mathbb{R}$, where $\mathcal{H}$ is an MD4-algebra.

II. If $\mathcal{G}$ is indecomposable, then we can choose a suitable basis $(X_1, X_2, X_3, X_4, X_5)$ of $\mathcal{G}$ such that $\mathcal{G}$ is isomorphic to one and only one of the following Lie algebra.

1. $\mathcal{G}^1 = \mathbb{R}.X_5 \equiv \mathbb{R}$.

$\mathcal{G}_{5,1} : [X_1, X_2] = [X_3, X_4] = X_5$; *the others Lie Brackets are trivial.*

2. $\mathcal{G}^1 = \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \equiv \mathbb{R}^2$

2.1. $\mathcal{G}_{5,2,1} : [X_1, X_2] = X_4,\ [X_2, X_3] = X_5$; *the others Lie brackets are trivial.*

2.2. $\mathcal{G}_{5,2,2(\lambda)} : [X_1, X_2] = [X_3, X_4] = X_5,\ [X_2, X_3] = \lambda X_4,$ $\lambda \in \mathbb{R}\backslash\{0\}$; *the others Lie Brackets are trivial.*

3. $\mathcal{G}^1 = \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \equiv \mathbb{R}^3, ad_{X_1} = 0,\ ad_{X_2} \in End(\mathcal{G}^1) \equiv Mat_3(\mathbb{R});\ [X_1, X_2] = X_3$.

3.1. $\mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$:

$$ad_{X_2} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{1\},\ \lambda_1 \neq \lambda_2 \neq 0.$$

3.2. $\mathcal{G}_{5,3,2(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

3.3. $\mathcal{G}_{5,3,3(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{1\}.$$

3.4. $\mathcal{G}_{5,3,4}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.5. $\mathcal{G}_{5,3,5(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{1\}.$$

3.6. $\mathcal{G}_{5,3,6(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

3.7. $\mathcal{G}_{5,3,7}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.8. $\mathcal{G}_{5,3,8(\lambda,\varphi)}$:

$$ad_{X_2} = \begin{pmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0\},\, \varphi \in (0, \pi).$$

4. $\mathcal{G}^1 = \mathbb{R}.X_3 \oplus \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \equiv \mathbb{R}^4$,

$$ad_{X_1} \in End(\mathcal{G}^1) \equiv Mat_4(\mathbb{R}).$$

4.1. $\mathcal{G}_{5,4,1(\lambda_1,\lambda_2,\lambda_3)}$:

$$ad_{X_1} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0, 1\}, \quad \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1.$$

4.2. $\mathcal{G}_{5,4,2(\lambda_1,\lambda_2)}$:

$$ad_{X_1} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0, 1\}, \lambda_1 \neq \lambda_2.$$

4.3. $\mathcal{G}_{5,4,3(\lambda)}$:

$$ad_{X_1} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

4.4. $\mathcal{G}_{5,4,4(\lambda)}$:

$$ad_{X_1} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

4.5. $\mathcal{G}_{5,4,5}$:

$$ad_{X_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4.6. $\mathcal{G}_{5,4,6(\lambda_1,\lambda_2)}$ *:*

$$ad_{X_1} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0, 1\}, \lambda_1 \neq \lambda_2.$$

4.7. $\mathcal{G}_{5,4,7(\lambda)}$:

$$ad_{X_1} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

4.8. $\mathcal{G}_{5,4,8(\lambda)}$:

$$ad_{X_1} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

4.9. $\mathcal{G}_{5,4,9(\lambda)}$:

$$ad_{X_1} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

4.10. $\mathcal{G}_{5,4,10}$:

$$ad_{X_1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

4.11. $\mathcal{G}_{5,4,11(\lambda_1,\lambda_2,\varphi)}$:

$$ad_{X_1} = \begin{pmatrix} cos\varphi & -sin\varphi & 0 & 0 \\ sin\varphi & cos\varphi & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix};$$

$$\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}, \lambda_1 \neq \lambda_2, \varphi \in (0, \pi).$$

4.12. $\mathcal{G}_{5,4,12(\lambda,\varphi)}$:

$$ad_{X_1} = \begin{pmatrix} cos\varphi & -sin\varphi & 0 & 0 \\ sin\varphi & cos\varphi & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi).$$

4.13. $\mathcal{G}_{5,4,13(\lambda,\varphi)}$:

$$ad_{X_1} = \begin{pmatrix} cos\varphi & -sin\varphi & 0 & 0 \\ sin\varphi & cos\varphi & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi).$$

4.14. $\mathcal{G}_{5,4,14(\lambda,\mu,\varphi)}$:

$$ad_{X_1} = \begin{pmatrix} cos\varphi & -sin\varphi & 0 & 0 \\ sin\varphi & cos\varphi & 0 & 0 \\ 0 & 0 & \lambda & -\mu \\ 0 & 0 & \mu & \lambda \end{pmatrix};$$

$$\lambda, \mu \in \mathbb{R}, \mu > 0, \varphi \in (0, \pi).$$

In proving Theorem 2.1, we need some lemmas.

Lemma 2.2. *For $X, Y \in \mathcal{G}\backslash\mathcal{G}^1$, $X \neq Y$, by considering ad_X, ad_Y as operators on $\mathcal{G}^1$ we have $ad_X \circ ad_Y = ad_Y \circ ad_X$.*

Proof. By using the Jacobi identity for X, Y and consider an arbitrary element $Z \in \mathcal{G}^1$, we have

$$\begin{aligned} & [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \\ \Leftrightarrow \quad & [X, [Y, Z]] - [Y, [X, Z]] = 0 \\ \Leftrightarrow \quad & ad_X \circ ad_Y(Z) = ad_Y \circ ad_X(Z); \ \forall Z \in \mathcal{G}^1 \\ \Leftrightarrow \quad & ad_X \circ ad_Y = ad_Y \circ ad_X. \end{aligned}$$

□

Lemma 2.3 (see [2, Chapter 2, Proposition 2.1])**.** *Let* $\mathcal{G}$ *be an MD-algebra with* $F \in \mathcal{G}^*$ *is not vanishing perfectly in* $\mathcal{G}^1$*, i.e. there exists* $U \in \mathcal{G}^1$ *such that* $\langle F, U \rangle \neq 0$*. Then the K-orbit* Ω_F *is one of the K-orbits having maximal dimension.*

Proof. Assume that Ω_F is not a K-orbit with maximal dimension, that is, $dim\Omega_F = 0$. Then we have

$$dim\mathcal{G}_F = dim\mathcal{G} - dim\Omega_F = dim\mathcal{G}.$$

Consequently, $KerB_F = \mathcal{G}_F = \mathcal{G} \supset \mathcal{G}^1$ and F is perfectly vanishing in $\mathcal{G}^1$. This contradicts the hypotheses of the Lemma. Therefore, Ω_F must be a K-orbit with maximal dimension. □

Lemma 2.4. *Let F be an arbitrary element of* $\mathcal{G}^*$*. Then* $dim\Omega_F = rank(B)$*, where* $B = (b_{ij})_5 := (\langle F, [X_j, X_i] \rangle), 1 \leq i, j \leq 5$*, is the matrix of the skew-symmetric bilinear form* B_F *in the basis* $(X_1, X_2, X_3, X_4, X_5)$ *of* $\mathcal{G}$*.*

Proof. Let $U = aX_1 + bX_2 + cX_3 + dX_4 + eX_5 \in \mathcal{G}$. Then we have

$$\begin{aligned} \mathcal{G}_F &= KerB_F \\ &= \{U \in \mathcal{G} / \langle F, [U, X_i] \rangle = 0; \, i = 1, 2, , 3, 4, 5\}. \end{aligned}$$

By simple computation, we obtain

$$U \in \mathcal{G}_F \Leftrightarrow B \begin{pmatrix} a \\ b \\ c \\ d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, $dim\Omega_F = dim\mathcal{G} - dim\mathcal{G}_F = rank(B)$. □

Lemma 2.5. *If $\mathcal{G}$ is a real solvable Lie algebra of dimension 5 with the first derived ideal $\mathcal{G}^1 \cong \mathbb{R}^4$ then $\mathcal{G}$ is a MD5-algebra.*

Proof. Let $\mathcal{G}$ be a real solvable Lie algebra with dimension 5 such that $\mathcal{G}^1$ is the commutative Lie algebra with dimension 4. Without loss of generality, we may assume that $\mathcal{G}^1 = \mathbb{R}.X_2 \oplus \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \equiv \mathbb{R}^4$, $ad_{X_1} = (a_{ij})_4 \in End(\mathcal{G}^1) \equiv Mat_4(\mathbb{R})$; $a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 4$.

Let $F = \alpha {X_1}^* + \beta {X_2}^* + \gamma {X_3}^* + \delta {X_4}^* + \sigma {X_5}^* \equiv (\alpha, \beta, \gamma, \delta, \sigma)$ be an arbitrary element from $\mathcal{G}^* \equiv \mathbb{R}^5; \alpha, \beta, \gamma, \delta, \sigma \in \mathbb{R}$. Then, by simple computation, we can see that the matrix B of the bilinear form B_F in the basis $(X_1, X_2, X_3, X_4, X_5)$ of $\mathcal{G}$ is a matrix of the following

$$\begin{pmatrix} 0 & -\sum_{i=2}^5 a_{i2}\alpha_i & -\sum_{i=2}^5 a_{i3}\alpha_i & -\sum_{i=2}^5 a_{i4}\alpha_i & -\sum_{i=2}^5 a_{i5}\alpha_i \\ \sum_{i=2}^5 a_{i2}\alpha_i & 0 & 0 & 0 & 0 \\ \sum_{i=2}^5 a_{i3}\alpha_i & 0 & 0 & 0 & 0 \\ \sum_{i=2}^5 a_{i4}\alpha_i & 0 & 0 & 0 & 0 \\ \sum_{i=2}^5 a_{i2}\alpha_i & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is now clear that $rank(B) \in \{0, 2\}$. Hence, according to Lemma 2.4, Ω_F is the orbit with dimension 0 or 2, i.e. $\mathcal{G}$ is an MD5-algebra. □

We now prove the main theorem.

Proof of Theorem 2.1.

It is clear that assertion I of Theorem 2.1 holds obviously. We only need to prove assertion II. Assume that $\mathcal{G}$ is an indecomposable MD5-algebra with basis $(X_1, X_2, X_3, X_4, X_5)$ and its first derived ideal $\mathcal{G}^1$ is commutative. Then $dim\, \mathcal{G}^1 \in \{1, 2, 3, 4\}$. In [16, Theorem 2.1] and [19, Theorem 3.2], the cases

had been considered when $dim\, \mathcal{G}^1 \in \{3, 4\}$. Therefore, we only need to consider the remaining cases when $dim\, \mathcal{G}^1 \in \{1, 2\}$. However, for the sake of completeness, we now consider here all cases.

1. $dim\mathcal{G}^1 = 1$. Without loss of generality , we may assume that $\mathcal{G}^1 = \mathbb{R}.X_5 \equiv \mathbb{R}$.

1.1. Assume that there exists $i \in \{1, 2, 3, 4\}$ with $[X_i, X_5] \neq 0$. Renumber the given basis, if necessary, and we suppose that $[X_4, X_5] = aX_5$, for some $a \in \mathbb{R}\backslash\{0\}$. Then, by changing X_4 with $X_4^{'} = \frac{1}{a}X_4$, we obtain $[X_4^{'}, X_5] = X_5$. Now, without any restriction of generality, we can assume that $[X_4, X_5] = X_5$.

Let $[X_i, X_5] = a_iX_5, [X_i, X_4] = b_iX_5; a_i, b_i \in \mathbb{R}; i = 1, 2, 3$. Then, by changing $X_i^{'} = X_i - a_iX_4 + b_iX_5 (i = 1, 2, 3)$, we get $[X_i^{'}, X_5] = [X_i^{'}, X_4] = 0; i = 1, 2, 3.$. Hence, we can always suppose right from the start that $[X_i, X_5] = [X_i, X_4] = 0; i = 1, 2, 3$.

Now, let $[X_i, X_j] = c_{ij}X_5$, $c_{ij} \in \mathbb{R}; 1 \leq i < j \leq 3$. Then, by using the Jacobi identity, we get $c_{ij} = 0$ for all i, j, $1 \leq i < j \leq 3$. But this shows that $\mathcal{G}$ is decomposable, which is a contradiction. Thus, this case cannot happen.

1.2. Assume that $[X_i, X_5] = 0$ for all i = 1, 2, 3, 4. Then, there exists $[X_i, X_j] = c_{ij}X_5$, $c_{ij} \neq 0$ for some $i, j \in \{1, 2, 3, 4\}, i \neq j$. By applying the same argument as in Case 1.1, we can suppose that $[X_1, X_2] = [X_3, X_4] = X_5$ and $[X_i, X_3] = [X_i, X_4] = 0; i = 1, 2$. Therefore, $\mathcal{G} \cong \mathcal{G}_{5,1}$.

2. $dim\mathcal{G}^1 = 2$. Without loss of generality, we now assume that $\mathcal{G}^1 = \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \equiv \mathbb{R}^2$; ad_{X_1}, ad_{X_2}, $ad_{X_3} \in End(\mathcal{G}^1) \equiv Mat_2(\mathbb{R})$.

2.1. $[X_i, X_j] = 0, 1 \leq i < j \leq 3$.

If there exists $ad_{X_i} = 0$ then $\mathcal{G}$ is decomposable, which is a contradiction. Hence, $ad_{X_i} \neq 0, i = 1, 2, 3$. We now show that we can always obtain $ad_{X_2} = 0$ by changing the basis. Indeed, we can let ad_{X_i} be $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $i = 1, 2, 3$. We first assume that $a_3 \neq 0$. Then, by writing $X_i^{'} = X_i - \frac{a_i}{a_3} X_3$, we get $ad_{X_i'} = \begin{pmatrix} 0 & b_i^{'} \\ c_i^{'} & d_i^{'} \end{pmatrix}$, $i = 1, 2$. Hence, we can suppose that $ad_{X_i} = \begin{pmatrix} 0 & b_i \\ c_i & d_i \end{pmatrix}$, $i = 1, 2$. According to Lemma 2.3, $ad_{X_1} \circ ad_{X_2} = ad_{X_2} \circ ad_{X_1}$. It follows that $ad_{X_1} = k.ad_{X_2}$, for some $k \in \mathbb{R} \backslash \{0\}$. By changing $X_2^{'} = X_2 - k.X_1$, we get $ad_{X_2'} = 0$, a contradiction. When $d_3 \neq 0$, then by using the same argument, we can also obtain a contradiction. Finally, assume that $a_3 = d_3 = 0$, $b_3^2 + c_3^2 \neq 0$. In view of Lemma 2.3, we get $ad_{X_i} \circ ad_{X_3} = ad_{X_3} \circ ad_{X_i} (i = 1, 2)$

Hence, it follows that $ad_{X_i} = k_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $0 \neq k_i \in \mathbb{R}$, $i = 1, 2$. In particular, $ad_{X_2} = k.ad_{X_1}, k = \frac{k_2}{k_1}$. Now, by changing $X_2^{'} = X_2 - k.X_1$, we get $ad_{X_2'} = 0$, again a contradiction. Hence, Case 2.1 can not happen.

2.2. Assume that there exists $[X_i, X_j] \neq 0, 1 \leq i < j \leq 3$ and $ad_{X_i} = 0$ $i = 1, 2, 3$.

It is clear that $\mathcal{G}^1 = \langle [X_1, X_2], [X_1, X_3], [X_2, X_3] \rangle$, and whence, the rank of $\{[X_1, X_2], [X_1, X_3], [X_2, X_3]\}$ is 2 and without restriction of generality, we can assume that $\{[X_1, X_2], [X_2, X_3]\}$ is a basis of $\mathcal{G}^1$. Let $[X_1, X_2] = aX_4 + bX_5$, $[X_2, X_3] = cX_4 + dX_5$ with $D := det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$. By changing basis as follows

$$X_4 = \frac{1}{D}(dX_4^{'} - bX_5^{'}),\ X_5 = \frac{1}{D}(-cX_4^{'} + aX_5^{'})$$

we get $[X_1, X_2] = X_4^{'}$, $[X_2, X_3] = X_5^{'}$. Hence, we can assume that $[X_1, X_2] = X_4$, $[X_2, X_3] = X_5$.

Let $[X_1, X_3] = \alpha X_4 + \beta X_5$. Then, by changing the basis as follows:

$$X_1^{'} = X_1 - \beta X_2, X_2^{'} = X_2, X_3^{'} = -\alpha X_2 + X_3$$

we get

$$[X_1^{'}, X_2^{'}] = X_4,\ [X_2^{'}, X_3^{'}] = X_5,\ [X_1^{'}, X_3^{'}] = 0.$$

Thus, we can always assume that

$$[X_1, X_2] = X_4,\ [X_2, X_3] = X_5,\ [X_1, X_3] = 0.$$

Therefore $\mathcal{G} \cong \mathcal{G}_{5,2,1}$.

2.3. Assume that there exists $[X_i, X_j] \neq 0$ and $ad_{X_k} \neq 0$, $1 \leq i \neq j \leq 3$, $1 \leq k \leq 3$. Then, without loss of generality, we may assume that $ad_{X_3} \neq 0$.

We can always change basis of $\mathcal{G}^1$ such that ad_{X_3} becomes one of the following matrices

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} cos\varphi & -sin\varphi \\ sin\varphi & cos\varphi \end{pmatrix}; \lambda \in \mathbb{R}, \varphi \in (0, \pi).$$

2.3a. Assume that $ad_{X_3} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then by using an argument analogous to that in Subsection 2.2, we get $ad_{X_1} = ad_{X_2} = 0$. Again, by Jacobi identity, we obtain $[X_1, X_2] = aX_5$, $a \in \mathbb{R}$.

Let $[X_i, X_3] = a_i X_4 + b_i X_5$; $a_i, b_i \in \mathbb{R}$, $i = 1, 2$. If $a = 0$, then by changing $X_i^{'} = X_i + b_i X_4$, we get $[X_i^{'}, X_3] = a_i X_4$, $i = 1, 2$. Hence, we can always assume from the outset that $[X_i, X_3] = a_i X_4$; $i = 1, 2$; ${a_1}^2 + {a_2}^2 \neq 0$. Without loss of generality, we may assume that $a_2 \neq 0$. Now, we change again the basis as follows

$$X_1^{'} = X_1 - \frac{a_1}{a_2} X_2,\ X_2^{'} = \frac{1}{a_2} X_2.$$

Then we get $[X_1{}', X_3] = 0, [X_2{}', X_3] = X_4$, i.e. $\mathcal{G}$ is decomposable, a contradiction. Hence, $a \neq 0$.

In the same way, we obtain

$$[X_1, X_2] = [X_3, X_4] = X_5,\ [X_2, X_3] = \lambda X_4,\ 0 \neq \lambda \in \mathbb{R}.$$

Therefore $\mathcal{G} \cong \mathcal{G}_{5,2,2(\lambda)}$.

2.3b. Assume that $ad_{X_3} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in \mathbb{R}$. Then, by using a similar argument as above, we get $\mathcal{G}_{5,2,3}$: $[X_1, X_2] = X_5, [X_3, X_4] = X_4$. By using Lemmas 2.2 and 2.3, and by direct computation, we can show that $\mathcal{G}_{5,2,3}$ is not an MD5-algebra. Hence, this case has to be rejected.

2.3c. Assume that $ad_{X_3} \in \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} cos\varphi & -sin\varphi \\ sin\varphi & cos\varphi \end{pmatrix}; \varphi \in (0, \pi) \right\}$. By using a similar argument as above, these cases have to be also rejected.

3. $dim\mathcal{G}^1 = 3$. We can always change basis to obtain $\mathcal{G}^1 = \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \equiv \mathbb{R}^3$; $ad_{X_1}, ad_{X_2} \in End(\mathcal{G}^1) \equiv Mat_3(\mathbb{R})$.

It is obvious that ad_{X_1} and ad_{X_2} cannot be the trivial operators concurrently because $\mathcal{G}^1 \cong \mathbb{R}^3$. Without loss of generality, we may assume that $ad_{X_2} \neq 0$. Then, by changing basis, if necessary, we obtain a similar classification of ad_{X_2} as follows

- $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{1\},\ \lambda_1 \neq \lambda_2 \neq 0);$

- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad (\lambda \in \mathbb{R} \setminus \{0,1\});$

- $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\lambda \in \mathbb{R} \setminus \{1\});$

- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$

- $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\lambda \in \mathbb{R} \setminus \{1\});$

- $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad (\lambda \in \mathbb{R} \setminus \{0,1\});$

- $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix};$

- $\begin{pmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad (\lambda \in \mathbb{R} \setminus \{0\}, \, \varphi \in (0,\pi)).$

Assume that $[X_1, X_2] = mX_3 + nX_4 + pX_5; m, n, p \in \mathbb{R}$. We can always change basis to have $[X_1, X_2] = mX_3$. Indeed, if

$$ad_{X_2} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \, (\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{1\}, \, \lambda_1 \neq \lambda_2 \neq 0),$$

then by changing X_1 for $X_1^{'} = X_1 + \frac{n}{\lambda_2}X_4 + pX_5$ we get $[X_1^{'}, X_2] = mX_3$, $m \in \mathbb{R}$. For the other values of ad_{X_2}, we can also change basis in the same way. Hence, without restriction of generality, we can assume that $[X_1, X_2] = mX_3$, $m \in \mathbb{R}$.

There are three cases which contradict each other as follows.

3.1. $[X_1, X_2] = 0$ (i.e. $m = 0$) and $ad_{X_1} = 0$. Then $\mathcal{G} = \mathcal{H} \oplus \mathbb{R}.X_1$, where $\mathcal{H}$ is the subalgebra of $\mathcal{G}$ generated by $\{X_2, X_3, X_4, X_5\}$, i.e. $\mathcal{G}$ is decomposable. Hence, this case is rejected.

3.2. $[X_1, X_2] = 0$ and $ad_{X_1} \neq 0$.

3.2a. Assume that $ad_{X_2} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{1\}$, $\lambda_1 \neq \lambda_2 \neq 0$. In view of Lemma 2.1, it follows by a direct computation that

$$ad_{X_1} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \xi \end{pmatrix} ; \mu, \nu, \xi \in \mathbb{R};\ \mu^2 + \nu^2 + \xi^2 \neq 0.$$

If $\xi \neq 0$, by changing $X_1^{'} = X_1 - \xi X_2$, we get

$$ad_{X_1^{'}} = \begin{pmatrix} \mu^{'} & 0 & 0 \\ 0 & \nu^{'} & 0 \\ 0 & 0 & 0 \end{pmatrix} ;$$

where $\mu^{'} = \mu - \xi\lambda_1, \nu^{'} = \nu - \xi\lambda_2$. Thus, we can assume that

$$ad_{X_1} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \mu, \nu \in \mathbb{R};\ \mu^2 + \nu^2 \neq 0.$$

Using Lemmas 2.2, 2.3, and by direct computation, we can show that $\mathcal{G}$ will not be an MD5-algebra in Case 3.2a . So this case must be rejected.

3.2b. In exactly the same way, but replacing the considered value of ad_{X_2} with the others, we can easily see that Case 3.2 cannot occur.

3.3. $[X_1, X_2] \neq 0$ (i.e. $m \neq 0$). By changing X_1 by $X_1{}' = \frac{1}{m}X_1$, we have $[X_1{}', X_2] = X_3$. Hence, without loss of generality, we may assume that $[X_1, X_2] = X_3$. By using a similar argument as the one in Case 3.2a, we obtain again a contradiction if $ad_{X_1} \neq 0$. In other words, $ad_{X_1} = 0$. Therefore, in the dependence on the value of ad_{X_2}, $\mathcal{G}$ must be isomorphic to one of the following algebras:

- $\mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$, $(\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{1\}, \lambda_1 \neq \lambda_2 \neq 0)$;
- $\mathcal{G}_{5,3,2(\lambda)}$, $(\lambda \in \mathbb{R} \setminus \{0, 1\})$;
- $\mathcal{G}_{5,3,3(\lambda)}$, $(\lambda \in \mathbb{R} \setminus \{1\})$;
- $\mathcal{G}_{5,3,4}$;
- $\mathcal{G}_{5,3,5(\lambda)}$, $(\lambda \in \mathbb{R} \setminus \{1\})$;
- $\mathcal{G}_{5,3,6(\lambda)}$, $(\lambda \in \mathbb{R} \setminus \{0, 1\})$;
- $\mathcal{G}_{5,3,7}$;
- $\mathcal{G}_{5,3,8(\lambda,\varphi)}$, $(\lambda \in \mathbb{R} \setminus \{0\}), \varphi \in (0, \pi)$).

Obviously, these algebras are not mutually isomorphic to each other.

4. $dim\mathcal{G}^1 = 4$. Without loss of generality, we may assume that $\mathcal{G}^1 = \mathbb{R}.X_2 \oplus \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \equiv \mathbb{R}^4$, $ad_{X_1} \in End(\mathcal{G}^1) \equiv Mat_4(\mathbb{R})$.

According to Lemma 2.5, the final assertions of Theorem 2.1 can be obtained by using similar classification of ad_{X_1}.

In view of Lemma 2.4, it follows by direct computation that all algebras listed in Theorem 2.1 are MD5-algebras. This completes the proof. □

Concluding Remark

Recall that every real Lie algebra $\mathcal{G}$ defines only one connected and simply connected Lie group G such that Lie(G) $= \mathcal{G}$. Therefore, we obtain a collection of twenty - five families of connected and simply connected MD5-groups corresponding to given indecomposable MD5-algebras in Theorem 2.1. For the sake of convenience, we denote every MD5-group from this collection by using the same indices as its corresponding MD5-algebra. For example, $G_{5,3,1(\lambda_1,\lambda_2)}$ is the connected and simply connected MD5-group which corresponds to $\mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$. All of these groups are indecomposable MD5-groups. In the next papers, we shall compute the invariants of given MD5-algebras, describe the geometry of K-orbits of its corresponding MD5-groups and also we shall classify topologically the MD5-foliations associated with these MD5-groups. In addition, characterization theorems of Connes C^*-algebras corresponding to these MD5-foliations will also be established.

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